On the fields of a torus and the role of the vector potential

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A toroidal current distribution has nonvanishing exterior vector potential **A**, but zero exterior field $\mathbf{B}=\nabla\times\mathbf{A}=0$. This property, together with the absence of fringing fields as in a cylindrical solenoid, makes it convenient for studies involving the vector potential in a field-free region, such as the Aharonov–Bohm effect, or the effect of **A** in a Josephson junction. We present an immediate general result of magnetostatics, and use it to easily compute **A** for a torus, to visualize the static vector potential for any current distribution, and to show how one can construct a current distribution to produce any desired **A**. When the torus current I varies in time, nonzero quasistatic fields $\mathbf{E}(t)$ and $\mathbf{B}(t)$ are produced ($\mathbf{E}\sim\omega I/r^3$ and $\mathbf{B}\sim\omega^2 I/r^2$). Radiation is also produced, with the radiation pattern of an *electric dipole*. The torus provides a counterexample to the common erroneous notion that if all multipole moments of a current distribution vanish then quasistatic fields and radiation must also vanish. We then formulate Maxwell's equations in a way that obviates the role of gauge transformations. This "gauge irrelevant" form clarifies the relation of potentials to current sources, isolating the role of the transverse part of **A**. The general result from magnetostatics is extended to time-varying sources, revealing a seldom recognized symmetry of Maxwell's equations, and showing how one can visualize **A** for an arbitrary time-dependent current source.

I. INTRODUCTION

In classical electromagnetic theory, the magnetic vector potential **A** is generally considered only a mathematical aid to solving for the field $\mathbf{B}=\nabla\times\mathbf{A}$. It is the field that is real.[1]

However, in classical physics energy is quite real, and the electrostatic potential ϕ is usually considered as real as the electric field $\mathbf{E}=-\nabla\phi$. Since relativity requires ϕ and **A** be components of a four vector, one should attribute just as much reality to **A**. Moreover, in quantum mechanics, the canonical quantization procedure *requires* the use of potentials. A problem would seem to arise in those situations in which $\mathbf{B}=0$, but $\mathbf{A}\neq 0$, for there should be no classical difference, but there may be a QM difference. In spite of these observations, the classical belief in the preeminent position of the fields held sway until nearly 1960.

Aharonov and Bohm[2] first pointed out experiments to demonstrate the reality and importance of the potentials in quantum mechanics. There are observable differences when, say, an electron is passed through a region of zero fields but nonzero potentials. The differences show up in the phase of the wave function, requiring an electron interference experiment to detect.

Ever since the reality of **A** in quantum mechanics was emphasized, and various experiments confirmed it,[3–7] interest has attached to measuring **A** directly,[8] especially where $\mathbf{B}=0$. Patents have been issued for a vector potential measuring device,[9] and a recent proposal suggests a nondestructive photon detector by "passively" measuring only **A** of the photon.[10] Indeed the possibility then arises of a class of electromagnetic sensors that would detect the vector potential rather than the fields.

To investigate this phenomenon, it is useful to have a ready current distribution that provides a working volume in which $\mathbf{B}=0$, but $\mathbf{A}\neq 0$. This is the case for the static **A** outside a torus on which flows a steady current only in the "toroidal" direction, around the thin limb. (But it is not the case for time varying currents.)

Two suggested detectors of **A** both operate via variants of the Aharonov–Bohm effect. In place of a coherent electron beam split into two sub-beams enclosing magnetic flux, one can use a superconductor with a Josephson junction. The superconducting state is a macroscopic coherent wave function that plays the role of the electron beam. The Josephson junction detects **A** since the tunneling current depends on the phase of the wave function, which shifts when $\mathbf{A}\neq 0$. This is the basis of patents by Gelinas.[9,11]

A second detector[10] would employ the conventional split electron beam along the surface of a crystal, detecting the evanescent **A** of a light beam undergoing total internal reflection.[12]

These sensors work best on a time varying **A**. Therefore for time dependent laboratory current sources, one needs to know the full environment at the detector, both fields and potentials.

A torus provides a simple source of nonzero **A** but zero **B** in a workable volume of space. In this note we first obtain a very useful expression for the static **A** outside a torus, and for the quasistatic and radiated fields and potential when the current varies in time. An Appendix presents the exact fields.

In the process we make useful observations in magnetostatics and electrodynamics concerning the calculation of the vector potential, and its relationship to sources in the light of gauge invariance. Maxwell's equations are formulated in a way that explicitly incorporates gauge invariance; once in that form questions of gauge transformations and gauge invariance never arise.

II. MAGNETOSTATICS

The equations of magnetostatics,

$$\nabla\cdot\mathbf{B}=0,\quad \nabla\times\mathbf{B}=\mu_0\mathbf{J},\tag{2.1}$$

are conveniently formulated with

$$\mathbf{B}\equiv\nabla\times\mathbf{A},\quad \nabla\cdot\mathbf{A}\ \text{arbitrary},\tag{2.2}$$

so that **A** obeys

$$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}.\tag{2.3}$$

The arbitrariness in $\nabla \cdot \mathbf{A}$ means the gradient of any scalar may be added to $\mathbf{A}$ without changing $\mathbf{B}$ (gauge transformation); it is sufficient to compute $\mathbf{A}$ in any gauge. Noting that in the static case the Lorentz ($\nabla \cdot \mathbf{A} + \partial\phi/\partial t = 0$) and Coulomb ($\nabla \cdot \mathbf{A} = 0$) gauges are the same, we choose

$$\nabla \cdot \mathbf{A} = 0. \tag{2.4}$$

Then $\mathbf{A}$ obeys Poisson's equation $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$, with the solution

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{2.5}$$

However the basic equations of $\mathbf{A}$,

$$\nabla \cdot \mathbf{A} = 0, \quad \nabla \times \mathbf{A} = \mathbf{B}, \tag{2.6}$$

are the same as obeyed by $\mathbf{B}$, Eqs. (2.1), with $\mathbf{A}$ replacing $\mathbf{B}$ and $\mathbf{B}$ replacing $\mu_0 \mathbf{J}$. Therefore $\mathbf{A}$ is the same function of $\mathbf{B}$ as $\mathbf{B}$ is of $\mu_0 \mathbf{J}$. We can introduce a vector Λ by

$$\mathbf{A} = \nabla \times \Lambda \tag{2.7}$$

and choose $\nabla \cdot \Lambda = 0$, so that

$$\Lambda(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{B}(\mathbf{r}')}{\mu_0 |\mathbf{r}-\mathbf{r}'|}. \tag{2.8}$$

Taking the curl of this shows $\mathbf{A}$ is given by a Biot–Savart law in terms of $\mathbf{B}$,

$$\mathbf{A}(\mathbf{r}) = \nabla \times \Lambda(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\mathbf{r}'-\mathbf{r}}{|\mathbf{r}-\mathbf{r}'|^3} \times \frac{\mathbf{B}(\mathbf{r}')}{\mu_0}. \tag{2.9}$$

Thus, as an immediate result of magnetostatics, $\mathbf{B}$ is the source of $\mathbf{A}$ just as $\mathbf{J}$ is the source of $\mathbf{B}$. This simple observation, often overlooked, can be used to visualize $\mathbf{A}$ in any situation, for $\mathbf{A}$ is constructed from $\mathbf{B}$ in the same way $\mathbf{B}$ is constructed from $\mathbf{J}$. One can infer features of $\mathbf{A}$ without having to carry out a calculation; for example, the "right-hand rule" determines $\mathbf{A}$'s direction from that of $\mathbf{B}$. And $\mathbf{A}$ can be computed quantitatively in two (identical) steps: first $\mathbf{B}$ from $\mathbf{J}$, then $\mathbf{A}$ from $\mathbf{B}$.

Magnetostatics admits an endless hierarchy of potentials,

$$\begin{aligned}
&\ldots\\
&\nabla \times \mathbf{J} = (\text{prescribed}) \quad \nabla \cdot \mathbf{J} = 0,\\
&\nabla \times \mathbf{B} = \mathbf{J} \quad \nabla \cdot \mathbf{B} = 0,\\
&\nabla \times \mathbf{A} = \mathbf{B} \quad \nabla \cdot \mathbf{A} = 0,\\
&\nabla \times \Lambda = \mathbf{A} \quad \nabla \cdot \Lambda = 0,\\
&\ldots
\end{aligned} \tag{2.10}$$

each determined from the next one by the curl operation. $\mathbf{B}$, for example, is the "potential" of the "field" $\mathbf{J}$, and $\mathbf{A}$ is the "field" of the "current" $\mathbf{B}$. This can be very useful in visualizing and computing the vector potential.

The hierarchy shows that to obtain the vector potential $\mathbf{A}_1$ of a current distribution $\mathbf{J}_1$ with field $\mathbf{B}_1$, we imagine a second current $\mathbf{J}_2$ equal to $\mathbf{B}_1$. Then $\mathbf{A}_1$ is equal to the field $\mathbf{B}_2$ of $\mathbf{J}_2$. For any $\mathbf{J}$ the solution for $\mathbf{B}$ immediately provides the solution for an infinite set of problems, and the hierarchy may be summarized as

$$\begin{pmatrix} \mathbf{J} \\ \mathbf{B} \\ \mathbf{A} \end{pmatrix}_{(n)} = \nabla \times \begin{pmatrix} \mathbf{J} \\ \mathbf{B} \\ \mathbf{A} \end{pmatrix}_{(n+1)}. \tag{2.11}$$

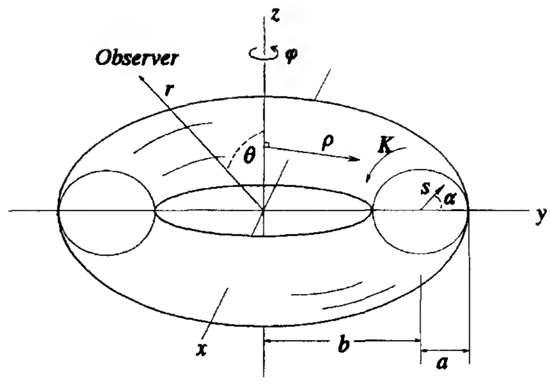

Fig. 1. Torus geometry definitions.

This is a general symmetry of magnetostatics. Use will be made of these observations in Sec. IX to construct a hierarchy of current distributions based on that of a torus. Equation (2.11) also allows easy construction of many current distributions which produce vanishing $\mathbf{B}$ but nonvanishing A. In Sec. X, it is extended to time-varying sources and fields.

III. STATIC VECTOR POTENTIAL OF TORUS

Direct evaluation of (2.5) for a torus runs into integrals of elliptic integrals. While these cannot be avoided for an exact solution by quadratures, the observations of Sec. II permit a very useful approximate expression.

The torus lies in the x,y plane, has minor radius a and major radius b (Fig. 1). To keep $\mathbf{B}=0$ outside, we require no circumferential current in the azimuthal direction about the symmetry axis z. The only allowed current is in the direction of increasing α, as would be created by a tightly spaced toroidal wire coil with an even number of counter-rotating layers. If i_w is the wire current, and the total number of turns is N, then the total current is $I = N i_w$.

We employ the usual spherical coordinate system (r,θ,φ), a Cartesian set of axes (x,y,z), the usual cylindrical system (ρ,φ,z), and occasionally the internal polar coordinates (s,α).

The field inside is

$$B = B_\varphi = \frac{\mu_0 I}{2\pi\rho}, \tag{3.1}$$

and decreases across the interior. The surface current density is $K = I/2\pi\rho = I/2\pi(b + a\cos\alpha)$ and the current density $\mathbf{J}$ is $K\delta(s-a)\hat{\alpha}$. The flux in the torus is

$$\Phi = \int d\mathbf{S} \cdot \mathbf{B} = \pi a^2 \frac{\mu_0 I}{2\pi b} g\left(\frac{a}{b}\right), \tag{3.2}$$

where $dS = s\,ds\,d\alpha$, and, with $u = s/a$, $\xi = a/b$,

$$g(\xi) = 2\int_0^1 du\, u \int_0^\pi \frac{d\alpha}{\pi} \frac{1}{1+u\xi\cos\alpha} = 2\,\frac{1-(1-\xi^2)^{1/2}}{\xi^2} \tag{3.3}$$

is the shape factor, $1 \leqslant g \leqslant 2$. $g \to 1$ as $a/b \to 0$ (bicycle tire), and $g \to 2$ as $a/b \to 1$ (holeless donut). Torus self inductance is $L = \mu_0 N^2 (a^2/2b) g$.

In the gauge $\nabla \cdot \mathbf{A} = 0$, the vector potential is given by the curl of (2.8),

$$\mathbf{A} = \nabla \times \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\hat{\varphi}' B(\mathbf{r}')}{\mu_0 |\mathbf{r}-\mathbf{r}'|}, \tag{3.4}$$

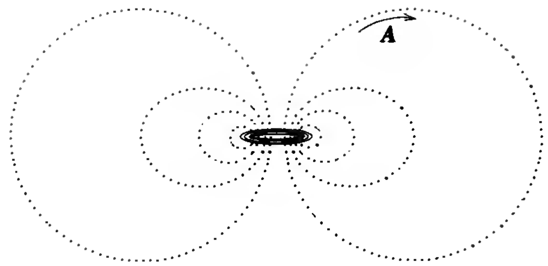

Fig. 2. Lines of **A**.

the integral being taken over the torus volume. **A** has no φ component. With B given by Eq. (3.1), this equation shows **A** is exactly the same as the magnetic field of a single fat wire loop coinciding with the torus with current distribution inside the wire proportional to $1/\rho$. When $b \gg a$, or when the observer is at $r \gg (b+a)$, this current variation within the wire is not important, and **A** looks like the dipole field of an ordinary current loop. Indeed the torus current density **J** is the curl of the current density of an ordinary loop (of wire radius a), so (2.11) applies.

The integral in Eq. (3.4) behaves as $1/r^2$ for $r \gg (b+a)$, and there is easily evaluated to obtain

$$\mathbf{A}_r = \frac{\mu_0}{4\pi} \frac{VI}{2\pi r^3} \cos\theta, \quad \mathbf{A}_\theta = \frac{\mu_0}{4\pi} \frac{VI}{4\pi r^3} \sin\theta \quad [r \gg (b+a)] \tag{3.5}$$

proportional to torus volume $V = 2\pi^2 a^2 b$. Lines of **A** are sketched in Fig. 2. The exact solution in Appendix A shows that **A** contains only odd powers of $1/r$. The next correction to the static potential (3.5) is $\sim 1/r^5$, smaller by a factor $\sim (b/r)^2$.

Outside the torus, **A** is locally the gradient of a scalar, $\mathbf{A} = -\nabla\Psi$ [for the $1/r^3$ terms, (3.5), Ψ is

$$\Psi = \frac{\mu_0}{4\pi} \frac{VI}{4\pi r^2} \cos\theta]. \tag{3.6}$$

Thus **A** can be locally transformed away with the gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\Psi = 0. \tag{3.7}$$

However **A** cannot be transformed to zero everywhere in the doubly connected space outside the torus. Due to (2.2), we have around any closed path encircling the limb,

$$\oint d\mathbf{l} \cdot \mathbf{A} = \Phi. \tag{3.8}$$

Therefore, on the surface, the average **A** around the limb is

$$\bar{A} = \frac{\Phi}{2\pi a} = \frac{\mu_0}{4\pi} \frac{aI}{b} g, \tag{3.9}$$

in any gauge. The largest $\bar{A}$ occurs for $a=b$, $\mathbf{A}_{\max} = (\mu_0/4\pi) 2I$.

In general, since $\mathbf{A} \propto \Phi$, but $\mathbf{B} \propto I$, **A** outside can be increased relative to any leakage **B** by keeping I constant and increasing a (so long as the gap between wires is not also increased).

The equivalence of **A** of a torus to **B** of a current loop becomes clear by dividing Eq. (3.4) by a, and using Eq. (3.1):

$$\frac{1}{a} \mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \int d^3 r' \frac{I \hat{\varphi}}{2\pi a \rho |\mathbf{r} - \mathbf{r}'|}. \tag{3.10}$$

A of a torus is a times the magnetic field of an ordinary loop with current density

$$\mathbf{J}' = \frac{I}{2\pi a \rho} \hat{\varphi} \tag{3.11}$$

and current

$$I' = \int dS\, J' = \left(\frac{a}{2b}\right) I g. \tag{3.12}$$

A. Potential on axis

On the z axis, **A** is simple when $a \ll b$. Then the magnetic field of a ring current $(a/2b)I$ is[13]

$$B_z = \frac{\mu_0}{4\pi} \frac{2\pi b^2 (a/2b) I}{(b^2+z^2)^{3/2}}, \tag{3.13}$$

so that $\mathbf{A}_z$ for a torus with $a \ll b$ and current I is

$$A_z = \frac{\mu_0}{4\pi} \frac{\pi a^2 b I}{(b^2+z^2)^{3/2}}. \tag{3.14}$$

$\mathbf{A}_z$ is largest at $z=0$, and there is quite comparable to the distant components equation (3.5) extrapolated back to $r \approx b$, but smaller by about a factor $\pi a/b$ than **A** on the surface.

B. The Aharonov–Bohm effect

Equation (3.8) is the important relation for the Aharonov–Bohm effect. The phase shift of an electron moving from point P to point Q is[2] $-(e/\hbar)\int_P^Q d\mathbf{l} \cdot \mathbf{A}$. An electron traversing a path 1, which passes through the torus, suffers a different phase shift $\delta\varphi$ from one traversing path 2, which does not, by a gauge invariant amount

$$\delta\varphi = -\left(\frac{e}{\hbar}\right) \oint_{(1,2)} d\mathbf{l} \cdot \mathbf{A} = -\left(\frac{e}{\hbar}\right) \Phi, \tag{3.15}$$

even though fields are zero along both paths. Therefore, the interference pattern at Q of a coherent electron source emanating from P, passing through the two slits with one torus arm between them, differs according as the magnetic flux between paths 1 and 2 is zero or nonzero, even though no force acts on an electron taking either path.

IV. ALTERNATE CONFIGURATION FOR EXPERIMENTS

Before proceeding to time varying currents, we note that the observations of Sec. II aid in designing other static configurations to produce a region of $\mathbf{B}=0$, but $\mathbf{A} \neq 0$.

Any localized current distribution $\mathbf{J}_1$ that produces a field $\mathbf{B}_1$ in a region where $\mathbf{J}_1 = 0$ can be transformed into a configuration that produces an $\mathbf{A} \neq 0$ where $\mathbf{B}=0$ by inventing a different current distribution that will produce a field equal to $\mathbf{J}_1$. According to the hierarchy of Eq. (2.11), the needed current distribution is proportional to $\nabla \times \mathbf{J}_1$. Since practical current sources are localized and produce a field outside themselves, this shows that, by constructing the new current distribution $\propto \nabla \times \mathbf{J}_1$, there are many ways to produce a field-free region of space with $\mathbf{A} \neq 0$.

The torus itself provides an example. The required current distribution is the curl of the torus current of Fig. 1, and is an

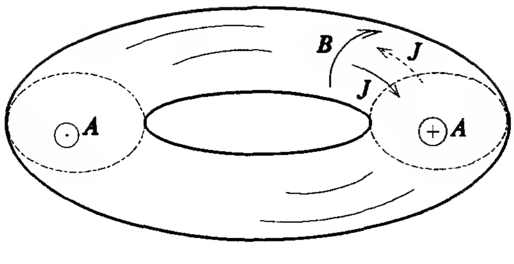

Fig. 3. Torus with counter-rotating surface current layers. The layer flowing in the $-\varphi$ direction is on the outer surface of the tube; the layer flowing in the $+\varphi$ direction is on the inner surface of the tube.

azimuthally flowing double layer on the torus surface. The inner surface layer, say, flows in the direction $\hat{\varphi}$, the outer surface layer in the direction $-\hat{\varphi}$. **B** is in a thin sheet confined between the layers, and is in the toroidal direction, being a vector parallel to the current layer K of Fig. 1. As shown in Fig. 3 the vector potential circulates inside the torus, like the **B** of Fig. 1. This would provide a volume of slowly varying **A**. Unfortunately, this volume is physically inaccessible.

However, if this torus is now cut through its limb at one place and then straightened out, we have a long narrow cylinder, of length $2\pi b$ and radius a, with current flowing up its length on the outside cylinder surface, and back down its length on the inside. The magnetic field is circumferential about the axis, confined between the two current sheets. The vector potential is the same as the usual magnetic field of a solenoid.

This configuration is itself topologically equivalent to the torus. Instead of changing the current direction and cutting the torus, we can stretch the torus of Fig. 1 in the z direction, its cross sectional circle of radius a being elongated into an ellipse, as in Fig. 4. After stretching to a length ℓ, and taking $a \to 0$, the resulting geometry is a "solenoid" of radius b and length ℓ, with azimuthal magnetic field confined between the two axial surface current sheets. $\mathbf{B}=0$ inside and outside this "solenoid." Inside, at cylindrical radii $\rho<b$, $\mathbf{A}\neq 0$; outside ($\rho>b$), **A** is only the "fringing field," significantly different from zero only near the ends. Inside the solenoid is a readily accessible volume for experiments.

A inside is proportional to the flux of **B**, and so can be increased by thickening the walls, as in the cylindrical torus of Fig. 5. **A** inside, on or off axis, is axial and is, for $\ell \gtrsim 2\rho_i$,

$$A=\frac{\mu_0 I}{2\pi}\ln\left(\frac{\rho_o}{\rho_i}\right). \tag{4.1}$$

Fig. 4. Showing topological equivalence of torus and cylindrical "solenoid" by stretching the torus vertically.

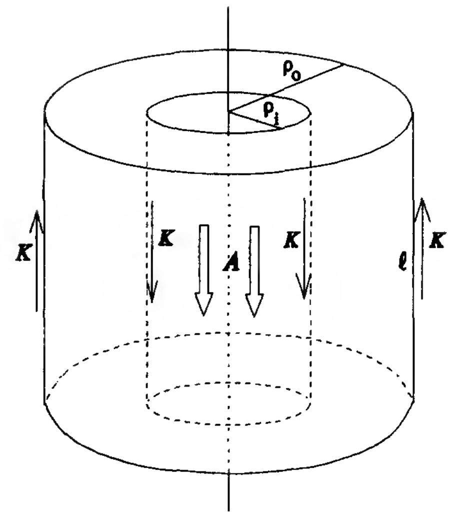

Fig. 5. Cylindrical torus with accessible region of large, constant **A** in the hole (at $\rho<\rho_i$).

A is maximized by increasing the ratio of outer to inner radius. For practical construction the logarithm is likely to be ~ 1, providing a useable volume with radius ρ_i of nearly constant **A** with magnitude as large as the largest occurring anywhere for any torus carrying the same current. If I varies in time, **B** always stays zero on axis, but $E_z=-\partial A_z/\partial t$ is nonzero, providing a (smaller) region near the axis of vanishing **B**, but nonzero $\mathbf{A}(t)$ and $\mathbf{E}(t)$.

The value for **A** in (4.1) is, of course, gauge dependent, here being in the gauge $\nabla\cdot\mathbf{A}=0$. As written this constant z component of **A** is $\nabla(zA)$, and so could be transformed away. But Eq. (3.8) requires that some **A** would then appear outside this cylindrical torus.

V. QUASISTATIC FIELDS

We return now to time dependent fields of a torus with circular cross section.

If **J** and I vary in time, an observer in the near zone will see the previously static **A** vary as $I(t)/r^3$ and give rise to an electric field. If, further, $\nabla\cdot\mathbf{J}$ remains zero, the scalar potential ϕ vanishes, and the field is $\mathbf{E}=-\partial\mathbf{A}/\partial t$. The quasistatic electric field pattern is the same as the static **A**, and, for harmonic variation with frequency ω, is proportional to $\omega I/r^3$. A displacement current $\epsilon_0\partial\mathbf{E}/\partial t\sim\omega^2 I/r^3$ appears through, say, the area enclosed in the circular path drawn above the torus in Fig. 6. Then Ampere's law

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}, \tag{5.1}$$

requires there be a quasistatic azimuthal magnetic field to balance this displacement current. Integrating Eq. (5.1) over the area of radius ρ, the required magnetic field is given by

$$2\pi\rho B=\frac{1}{c^2}\frac{\partial}{\partial t}\int d\mathbf{S}\cdot\mathbf{E}=\frac{2\pi}{c^2}\frac{\partial}{\partial t}\int d\rho\,\rho E_z. \tag{5.2}$$

Since $E_z\sim\omega I/\rho^3$ for large ρ, this quasistatic **B** varies as $\omega^2 I/r^2$ for large r.

This **B** must also be $\nabla\times\mathbf{A}$. But the just determined quasistatic **B** is not the curl of the quasistatic **A**, which vanishes.

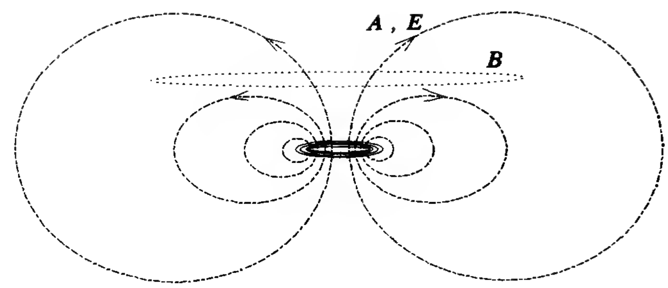

Fig. 6. A quasistatic azimuthal magnetic field is required on the indicated loop due to the displacement current through it.

The field discussion thus far therefore cannot be complete, and we need to look more closely at time-varying fields and potentials.

In the Lorentz gauge, Maxwell's equations reduce to the wave equation for the vector potential, whose Cartesian components are given exactly by the full retarded solution

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\int d^3\mathbf{r}'\ \frac{\mathbf{J}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|},\tag{5.3}$$

where

$$t'=t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}\tag{5.4}$$

is retarded time. When $\mathbf{A}$ is expanded in powers of b/r, and each coefficient further developed in the low frequency expansion in powers of $kb\equiv(\omega/c)b$, the lowest-order surviving terms are given in Appendix A, Eq. (A17). The quasistatic vector potential is

$$\mathbf{A}_{\mathrm{QS}}=e^{-i\omega t}e^{ikr}\mathbf{A}_{\mathrm{stat}}(\mathbf{r}),\tag{5.5}$$

where $\mathbf{A}_{\mathrm{stat}}$ is the static vector potential of Eq. (3.5).

The quasistatic fields (lowest order in frequency) are

$$\mathbf{E}_{\mathrm{QS}}=\frac{\mu_0}{4\pi}\frac{VI}{4\pi}\frac{ikc}{r^3}\,(2\cos\theta\hat{r}+\sin\theta\hat{\theta}),$$

$$\mathbf{B}_{\mathrm{QS}}=\frac{\mu_0}{4\pi}\frac{VI}{4\pi}\frac{k^2}{r^2}\sin\theta\hat{\varphi},\tag{5.6}$$

also proportional to torus volume. The quasistatic $\mathbf{B}$ does balance the displacement current as required by Ampere's law, $\partial\mathbf{E}_{\mathrm{QS}}/\partial t=c^2\nabla\times\mathbf{B}_{\mathrm{QS}}$. As seen in the Appendix, $\mathbf{E}_{\mathrm{QS}}$ arises from the time derivative of the quasistatic potential $\mathbf{A}_{\mathrm{QS}}$, but $\mathbf{B}_{\mathrm{QS}}$ arises as the curl of the "inductive" $\mathbf{A}$ which is $-ikr$ times $\mathbf{A}_{\mathrm{QS}}$. However, these quasistatic fields do not separately satisfy Faraday's law, $\partial\mathbf{B}_{\mathrm{QS}}/\partial t\neq-\nabla\times\mathbf{E}_{\mathrm{QS}}$. Rather, $\nabla\times\mathbf{E}_{\mathrm{QS}}=0$, and $\partial\mathbf{B}_{\mathrm{QS}}/\partial t$ is balanced by the curl of the part of $\mathbf{E}$ arising from the inductive $\mathbf{A}$, which is kr times smaller than $\mathbf{E}_{\mathrm{QS}}$.

These fields, valid for $kr\ll1$, are ordered according to

$$B_{\mathrm{QS}}\sim\frac{VIk^2}{r^2}\sim kr\,\frac{E_{\mathrm{QS}}}{c}\ll\frac{E_{\mathrm{QS}}}{c}.\tag{5.7}$$

For most charge–current distributions ρ, $\mathbf{J}$, the quasistatic fields are simply the static fields with $\rho(t)$, and $\mathbf{J}(t)$ in place of their static counterparts. This is decidedly not the case for a torus.

VI. RADIATED FIELDS AND POTENTIAL

Before computing the radiation fields from a torus we make some general observations.

A. General comments on radiation

In any gauge, the scalar potential is not needed for the radiated fields, since $\mathbf{B}=\nabla\times\mathbf{A}$, and $\mathbf{E}_{\mathrm{rad}}=-c\mathbf{n}\times\mathbf{B}_{\mathrm{rad}}$, where $\mathbf{n}$ is the outgoing unit vector. In the Lorentz gauge, $\mathbf{A}$ is given by (5.3). The radiated vector potential is the part that falls as $1/r$, obtained by expanding $1/|\mathbf{r}-\mathbf{r}'|$ and keeping only the $1/r$ term,

$$\mathbf{A}_{\mathrm{rad}}=\frac{\mu_0}{4\pi r}\int d^3r'\ \mathbf{J}(\mathbf{r}',t').\tag{6.1}$$

If the time argument of $\mathbf{J}$ were t, the integral of $\mathbf{J}$ over all space at one instant of time would vanish when $\nabla\cdot\mathbf{J}=0$. Retarded time means that wavelets emanating from different parts of the source do not quite cancel, allowing radiation. Retardation, of course, is the physical reason any divergenceless time-dependent current distribution of finite extent radiates.

$\mathbf{B}_{\mathrm{rad}}$ is the $1/r$ part of $\nabla\times\mathbf{A}_{\mathrm{rad}}$, or

$$\mathbf{B}_{\mathrm{rad}}=\frac{\mu_0}{4\pi r}\nabla\times\int d^3r'\ \mathbf{J}(\mathbf{r}',t').\tag{6.2}$$

This does not vanish even for a torus. The integral depends on $\mathbf{r}$ only through t',

$$\nabla\times\mathbf{J}(\mathbf{r}',t')=\nabla t'\times\frac{\partial\mathbf{J}}{\partial t'}=-\frac{1}{c}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\times\frac{\partial\mathbf{J}}{\partial t'},\tag{6.3}$$

where we have used the gradient of (5.4). Therefore,

$$\mathbf{B}_{\mathrm{rad}}=-\frac{\mu_0}{4\pi cr}\int d^3r'\ \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\times\frac{\partial\mathbf{J}(\mathbf{r}',t')}{\partial t'}.\tag{6.4}$$

The first factor in this integrand is $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|=\mathbf{n}-O(r'/r)$, so that

$$\mathbf{B}_{\mathrm{rad}}=-\frac{\mu_0}{4\pi cr}\mathbf{n}\times\int d^3r'\ \frac{\partial\mathbf{J}}{\partial t'}.\tag{6.5}$$

Now noting $\partial t'/\partial t=1$, the derivative may be pulled out,

$$\mathbf{B}_{\mathrm{rad}}=-\frac{\mu_0}{4\pi cr}\mathbf{n}\times\frac{\partial}{\partial t}\int d^3r'\mathbf{J}(\mathbf{r}',t')=$$

$$-\frac{1}{c}\mathbf{n}\times\frac{\partial\mathbf{A}_{\mathrm{rad}}}{\partial t}.\tag{6.6}$$

These equations also show $\mathbf{E}_{\mathrm{rad}}=-\partial\mathbf{A}_{\mathrm{rad}}/\partial t$ as it should.

In any gauge, if $\mathbf{A}_{\mathrm{rad}}$ is nonzero, $\mathbf{B}_{\mathrm{rad}}$ and $\mathbf{E}_{\mathrm{rad}}$ are also nonzero. Unlike the static $\mathbf{A}$, the radiated vector potential cannot, of course, be separated from its fields. That is, there is no such thing as a "radiated curl-free vector potential" occasionally referred to. It is easily shown that $\nabla\cdot\mathbf{A}_{\mathrm{rad}}=0$, so that $\mathbf{A}_{\mathrm{rad}}$ is a transverse vector.

The discussion so far is completely general, applying to any current distribution.

As a rule, if a current increases in time from zero to a nonzero steady value, radiation is produced that leaves behind the static field of the nonzero current. When I was turned on for the torus, radiated $\mathbf{A}$, $\mathbf{E}$, and $\mathbf{B}$ propagated out, leaving behind a nonzero static $\mathbf{A}$, and a static $\mathbf{B}$ which happens to have value 0. The static $\mathbf{A}$ outside a torus is only accidentally curl-free because of high-symmetry geometry.

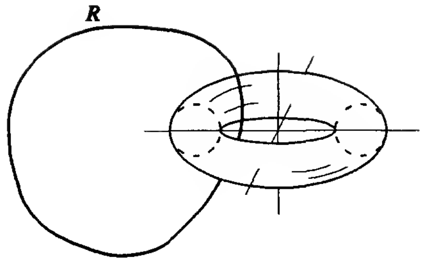

Fig. 7. Resistive wire loop through a torus.

As shown in Sec. VII it is more fundamentally to be considered a transverse vector, with the defining property of being divergence-free everywhere.

B. Physical argument for radiation

It is instructive to understand physically why a system that encloses magnetic flux in an area that can be "looped" must radiate. Consider a resistive test wire that encircles the torus limb in a closed loop as in Fig. 7. Let R be the full resistance of the wire. When the torus current I varies, so does the magnetic flux Φ in the torus interior, and an emf

$$\mathscr{E}=\oint_{\text{wire}} d\mathbf{l}\cdot\mathbf{E}=-\frac{\partial\Phi}{\partial t} \tag{6.7}$$

is produced, according to Faraday's law. A current $\mathscr{E}/R$ flows in the test wire. The electrons in the wire know to move because the induced electric field drives them.

All electromagnetic disturbances start where $\nabla\times\mathbf{J}\neq 0$, in this case on the torus windings. The only way $\mathbf{E}$ can get to the wire is by propagating from the torus to the wire. This propagation, carried to larger distances, is radiation.

If the torus and test wire are of very great radius L, then a propagation time of order L/c is necessary before $\mathbf{E}$ gets to the wire. But since Eq. (6.7) holds instantaneously, the radiated magnetic flux through the big wire loop is equal and opposite to the quasistatic flux inside the torus limb until the radiation front gets to the wire. It is only after the radiated fields pass that Φ in Eq. (6.7) can be taken to be the usual interior quasistatic flux. Only $1/r$ fields can account for the necessary emf. Radiation from a torus has been discussed by Baum.[14]

It is interesting to note that, from Eq. (6.7), and from Eq. (3.8), which holds for time varying conditions as well, the wire current is

$$\frac{\mathscr{E}}{R}=\frac{1}{R}\oint d\mathbf{l}\cdot\mathbf{E}=-\frac{1}{R}\oint_{\text{wire}} d\mathbf{l}\cdot\frac{\partial\mathbf{A}}{\partial t}, \tag{6.8}$$

so the total charge moved through the wire is

$$Q(t)=\int\frac{\mathscr{E}(t)}{R}\,dt=-\frac{1}{R}\oint d\mathbf{l}\cdot\mathbf{A}. \tag{6.9}$$

That is, displaced charge is as much a *direct* measure of $\oint d\mathbf{l}\cdot\mathbf{A}$ as current is of $\oint d\mathbf{l}\cdot\mathbf{E}$. From this point of view $\mathbf{A}$ could be considered just as real a physical variable as $\mathbf{E}$. Feynman[15] stresses how $\mathbf{A}$ may be considered physically real in spite of the arbitrariness in its divergence.

Parallel to the fact that only $\nabla\times\mathbf{A}$ enters Maxwell's equations and $\nabla\cdot\mathbf{A}$ is arbitrary, it is worth noting that only the divergence of the energy flux $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ (Poynting vector) enters the energy conservation law that follows from Maxwell's equations; $\nabla\times\mathbf{S}$ is arbitrary. $\mathbf{S}$ itself is arbitrary to the extent that the curl of any vector may be added to it. Yet we are quite accustomed to thinking of $\mathbf{S}$ as physically real. An arbitrary additive curl or gradient need not prevent a vector from being considered real. Konopinski[16] discusses how in classical electromagnetism $\mathbf{A}$ may be considered the potential momentum per unit charge of a particle in external fields, just as ϕ is the potential energy per unit charge.

C. Explicit radiated fields

To compute the radiated fields, we directly evaluate Eq. (6.1) for a harmonic toroidal current. For frequency ω, $\mathbf{J}$ is

$$\mathbf{J}(\mathbf{r},t')=\frac{I}{2\pi\rho}\,\delta(s-a)e^{-i\omega t'}\,\hat{\alpha}, \tag{6.10}$$

where $\hat{\alpha}$ is the unit vector in the direction of increasing α (Fig. 1). In Eq. (6.1) the s integral in $d^3r'=\rho'\,d\varphi'\,ds\,s\,d\alpha$ is trivial, leaving

$$\mathbf{A}_{\text{rad}}(\mathbf{r}',t)=\frac{\mu_0}{4\pi}\frac{Ia}{2\pi r}\int_0^{2\pi}d\varphi'\int_0^{2\pi}d\alpha\,\hat{\alpha}\,e^{-i\omega t'}. \tag{6.11}$$

In t', $|\mathbf{r}-\mathbf{r}'|$ is expanded

$$t'=t-\frac{r}{c}+\frac{1}{c}\,\mathbf{n}\cdot\mathbf{r}'+O(r'^2/cr) \tag{6.12}$$

so the integral in Eq. (6.11) becomes

$$e^{-i\omega\tau}\int d\varphi'\int d\alpha\,\hat{\boldsymbol{\alpha}}\,e^{-i\mathbf{k}\cdot\mathbf{r}'}, \tag{6.13}$$

where $\tau=t-r/c$ is the observer's retarded time, and $\mathbf{k}=(\omega/c)\mathbf{n}$. We have

$$\hat{\boldsymbol{\alpha}}=-(\hat{x}\cos\varphi'+\hat{y}\sin\varphi')\sin\alpha+\hat{z}\cos\alpha \tag{6.14}$$

and

$$\mathbf{k}\cdot\mathbf{r}'=k[(b+a\cos\alpha)\cos(\varphi-\varphi')\sin\theta+a\sin\alpha\cos\theta], \tag{6.15}$$

where θ,φ are those of the observer.

Equations (6.14) and (6.15) are to be inserted in Eq. (6.13). Rather than grapple with the resultant integrals we proceed with a low-frequency approximation, $kb\ll 1$. When the exponential in the integrand in Eq. (6.13) is expanded, $\exp(-i\mathbf{k}\cdot\mathbf{r}')=1-i\mathbf{k}\cdot\mathbf{r}'-(\mathbf{k}\cdot\mathbf{r}')^2/2\pm\ldots$, it is easy to show the first two terms do not contribute to Eq. (6.13), and the remaining integral is trivial. One finds,

$$\begin{aligned}\int_0^{2\pi}d\varphi'\int_0^{2\pi}d\alpha\,\hat{\boldsymbol{\alpha}}e^{-i\mathbf{k}\cdot\mathbf{r}'}&=\pi^2k^2ab\sin\theta(\hat{x}\cos\theta\cos\varphi+\hat{y}\cos\theta\sin\varphi\\&\quad-\hat{z}\sin\theta)=\pi^2k^2ab\sin\theta\hat{\theta}.\end{aligned} \tag{6.16}$$

Then, if we call the time-dependent current I_{AC}, to lowest nonvanishing order in kb the radiated $\mathbf{A}$ of a torus is, except for phase,

$$\mathbf{A}_{\text{rad}}=\frac{\mu_0}{4\pi}\frac{k^2I_{\text{AC}}V}{4\pi r}\sin\theta\hat{\theta}, \tag{6.17}$$

a factor of order $(I_{\text{AC}}/I)(kr)^2$ times the static $\mathbf{A}$ equation (3.5) in the same (Lorentz) gauge. Fields are

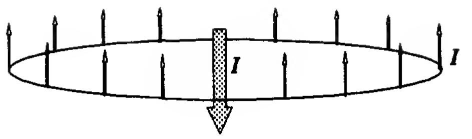

Fig. 8. Exaggerated model of vertical torus currents.

$$\mathbf{B}_{\text{rad}}=\frac{\mu_0}{4\pi}\frac{k^3 I_{\text{AC}} V}{4\pi r}\sin\theta\hat{\varphi},$$

$$\mathbf{E}_{\text{rad}}=\frac{\mu_0}{4\pi}\frac{ck^3 I_{\text{AC}} V}{4\pi r}\sin\theta\hat{\theta}=-c\hat{r}\times\mathbf{B}_{\text{rad}}. \tag{6.18}$$

Radiated field amplitude is proportional to torus volume. The field pattern is that of an *electric dipole*. However the fields are proportional to ω^3 rather than ω^2. Physically, the origin of the fields is as follows. The torus of Fig. 1 has a current I flowing in the $+z$ direction at $\rho=b+a$, and I flowing in the $-z$ direction at $\rho=b-a$, with connecting paths above and below. Exaggerating the radii, Fig. 8 sketches a ring of upward flowing current balanced by a downward flowing current on axis. If I varies in time, both the center and the outer ring currents will radiate like individual electric dipoles. The electric dipole moment of the entire configuration is zero, since the charge displaced by one current is taken by the other. Since the sources are of different dimensions, their radiated fields do not cancel. Rather, the net field is proportional to ka, the separation relative to a wavelength. This explains the extra power of ω and of a in the radiated fields relative to an ordinary electric dipole.

One is reminded that even two simple dipoles, separated by d as in Fig. 9, together having zero dipole moment, will radiate an electric dipole field at frequencies $\omega \gtrsim c/d$. At lower frequencies the quadrupole field dominates, the electric dipole part being smaller by a factor of order $\omega d/c$.

D. Multipole coefficients

It is curious that all electric and magnetic multipole moments of a toroidal current distribution vanish, and yet quasistatic fields and radiation do not.

The general theory of multipole radiation [e.g., Jackson;[17] Chap. 16] relates radiated fields to sources. The relevant source parameters are the electric and magnetic *multipole coefficients*, different from multipole moments in that they account for retardation across the source. Multipole moments form a complete set for static charge-current distributions, multipole coefficients for the time varying case.

All magnetic multipole coefficients vanish for the torus, but the electric coefficients are nonzero. The electric coefficients $a_E(l,m)$ are given in Jackson,[17] Eq. (16.91), and contain a contribution from charge density ρ, current density $\mathbf{J}$, and intrinsic magnetization $\mathbf{M}$. The term in ρ corresponds to the usual electric multipole moment when $kr \ll 1$ [generalized to include the radial radiation function $j_l(kr)$], and vanishes for a torus, as does the term in $\mathbf{M}$.

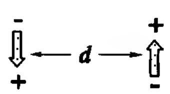

Fig. 9. Two nearby small dipoles.

The term in $\mathbf{J}$ is not zero. The extra factor kr in this term accounts for the destructive interference from opposite sides of closed loop currents. The lowest coefficient is $l=1$, $m=0$, and corresponds precisely to the previous explanation in terms of equal up and down currents at different radii. It is readily evaluated for the case $kb \ll 1$, and results in a radiated field equal to our Eq. (6.18).

The general expression for $a_E(l,m)$ shows that a system with $\rho=\mathbf{J}=0$, but with nonvanishing time varying $\mathbf{M}$, also produces electric dipole radiation.

There is no general formalism expressing quasistatic fields in terms of source moments as there is for static and radiation fields. By explicit calculation one finds quasistatic fields of a torus are nonzero and given by Eqs. (5.6).

VII. GAUGE TRANSFORMATIONS AND SOURCES

Due to the arbitrariness arising from a gauge transformation, only certain parts of $\mathbf{A}$ are in principle measurable. One needs to inquire how $\mathbf{A}$ is related to current sources, since classically the sole *raison d'etre* of $\mathbf{A}$ is to have its curl equal to $\mathbf{B}$. The longitudinal part of $\mathbf{A}$ can be combined with the scalar potential ϕ, while the transverse part retains its identity and is measurable.

The source-free Maxwell equations (Faraday's law, and the absence of magnetic monopoles) serve to define the fields in terms of the potentials, and the two equations with sources (Ampere's and Gauss' laws) become dynamical equations determining the potentials. If we make the transformation

$$\mathbf{A}_1 \to \mathbf{A}_2 = \mathbf{A}_1 + \nabla\chi,$$

$$\phi_1 \to \phi_2 = \phi_1 - \dot{\chi}, \tag{7.1}$$

where χ is an arbitrary function, gauge invariance assures that χ drops out of the dynamical equations altogether and has no physical meaning.

A. Transverse and longitudinal fields

The relation $\nabla\times\mathbf{A}=\mathbf{B}$ and the arbitrariness in $\nabla\cdot\mathbf{A}$ suggest writing all fields in their transverse (solenoidal) and longitudinal (irrotational) parts.[18] Thus, for $\mathbf{A}$,

$$\begin{aligned}&\mathbf{A}=\mathbf{A}_T+\mathbf{A}_L,\\ &\nabla\times\mathbf{A}_T=\nabla\times\mathbf{A}=\mathbf{B},\quad \nabla\times\mathbf{A}_L\equiv 0,\\ &\nabla\cdot\mathbf{A}_T\equiv 0,\quad \nabla\cdot\mathbf{A}_L=\nabla\cdot\mathbf{A}=\text{arbitrary}.\end{aligned} \tag{7.2}$$

If two vectors are equal, their transverse and longitudinal parts are separately equal. Then, decomposing $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{J}$ into their parts, Maxwell's equations can be written

$$\begin{aligned}&\mathbf{B}_L=0,\quad \nabla\times\mathbf{E}_T+\frac{\partial\mathbf{B}_T}{\partial t}=0;\\ &\nabla\cdot\mathbf{E}_L=\rho/\epsilon_0,\\ &\nabla\times\mathbf{B}_T-\frac{1}{c^2}\frac{\partial\mathbf{E}_T}{\partial t}=\mu_0\mathbf{J}_T,\qquad -\frac{1}{c^2}\frac{\partial\mathbf{E}_L}{\partial t}=\mu_0\mathbf{J}_L;\end{aligned} \tag{7.3}$$

with charge conservation

$$\nabla\cdot\mathbf{J}_L+\frac{\partial\rho}{\partial t}=0. \tag{7.4}$$

B. Maxwell's equations without gauge transformations

Now introduce the usual potentials $\mathbf{A}$ and ϕ:

$$\mathbf{B}_T=\nabla\times\mathbf{A}_T;$$

$$\mathbf{E}_T=-\frac{\partial\mathbf{A}_T}{\partial t},\quad \mathbf{E}_L=-\frac{\partial\mathbf{A}_L}{\partial t}-\nabla\phi. \tag{7.5}$$

Only $\mathbf{A}_L$ and ϕ are affected by a gauge transformation, with $\mathbf{E}_L$ left unchanged. $\mathbf{A}_L$ can always be written in terms of a scalar χ_0, $\mathbf{A}_L=\nabla\chi_0$, so that $\mathbf{E}_L$ can be expressed as

$$\mathbf{E}_L=-\nabla\psi, \tag{7.6}$$

where

$$\psi=\phi+\dot{\chi}_0 \tag{7.7}$$

is a *gauge invariant scalar potential* with only an additive constant arbitrary. By itself $\mathbf{A}_L$ is physically irrelevant, being a matter of gauge choice and having no relation to sources. It can be fully subsumed in the definition of the scalar potential in a gauge invariant way, and only ψ and $\mathbf{A}_T$ survive. In terms of them Maxwell's equations are

$$\mathbf{B}_T=\nabla\times\mathbf{A}_T,\quad \mathbf{B}_L=0;$$

$$\mathbf{E}_T=-\frac{\partial\mathbf{A}_T}{\partial t},\quad \mathbf{E}_L=-\nabla\psi;$$

$$\nabla^2\psi=-\frac{\rho}{\epsilon_0};$$

$$\Box\mathbf{A}_T=-\mu_0\mathbf{J}_T,\quad \frac{\partial}{\partial t}\nabla\psi=\frac{\mathbf{J}_L}{\epsilon_0}, \tag{7.8}$$

where $\Box=\nabla^2-(1/c^2)\partial^2/\partial t^2$.

Since $\mathbf{A}_L$ has been combined with ϕ, the last equation for ψ is redundant and may be dropped. It is equivalent to the remaining Poisson equation and the continuity equation (7.4). The Poisson equation for ψ, and wave equation for $\mathbf{A}_T$, are the same as the usual equations for ϕ and $\mathbf{A}$ in the Coulomb gauge, with the significant difference that now there has been no specification of gauge. Every quantity in Eq. (7.8) is gauge independent.

With Maxwell's equations in this form there is no longer any arbitrariness in the potentials, and the questions of gauge transformations or gauge invariance do not arise. Further, longitudinal sources, fields, and potentials completely decouple from transverse ones.

Maxwell's equations are usually not formulated this way. Most theories insist on, and build in, relativistic invariance, and separately inquire as to gauge invariance. In contrast, the above equations build in gauge invariance (indeed, gauge irrelevance), but are not manifestly Lorentz invariant. The very division into transverse and longitudinal parts is not Lorentz invariant. However the formulation can be convenient for analysis of a given experiment in one reference frame.

$\mathbf{J}_L$ gives rise only to ψ. $\mathbf{J}_T$ gives rise only to $\mathbf{A}_T$. Closed current loops, for example, which are commonly well approximated as divergenceless, especially at low frequencies, are transverse currents, and produce only $\mathbf{A}_T$ and transverse fields. In this same approximation, the complete fields of a torus with time-dependent current are purely transverse.

The discussion following Eq. (6.6), that $\mathbf{A}_{\text{rad}}$ cannot be separated from its fields, left open the possibility that the longitudinal part of $\mathbf{A}$ might be separable from its fields. But $\mathbf{A}_L$, and the field associated with it, is a gauge artifice that is eliminated in Eq. (7.8), and has no physical meaning.

The static $\mathbf{A}$ of a torus, Eq. (3.5), is related to the source current. This $\mathbf{A}$ is a transverse vector for which $\nabla\cdot\mathbf{A}=0$ everywhere and which *happens* to have $\nabla\times\mathbf{A}=0$ outside the torus (but $\nabla\times\mathbf{A}\neq 0$ inside). It is *locally* curl-free due to high symmetry.

These equations make it clear that it is only the transverse part of the current source that produces radiation. This same result is apparent from the usual wave equation for $\mathbf{B}$

$$\Box\mathbf{B}=-\mu_0\nabla\times\mathbf{J}=-\mu_0\nabla\times\mathbf{J}_T \tag{7.9}$$

that follows from Maxwell's equations.

VIII. CONDUCTORS AS SHIELDS

We inquire as to the effectiveness of conducting enclosures in shielding the vector potential.

A. Boundary conditions on A

Boundary conditions on $\mathbf{A}$ are conventionally derived[19] by applying the usual Gaussian pill box or Stokes loop over an interface between two media. But they can be obtained more directly from those on $\mathbf{B}$ by using the symmetry observed in Sec. II.

Since the normal component $\mathbf{B}_\perp$ of $\mathbf{B}$ is continuous at an interface, and since this $\mathbf{B}$ is the $\mathbf{A}$ of a different problem, the normal component $\mathbf{A}_\perp$ of $\mathbf{A}$ is continuous also. Actually, for the general case, the discontinuity in $\mathbf{A}_\perp$ is formally gauge-dependent, since $\nabla\cdot\mathbf{A}$ need not vanish at the surface. But so long as the gauge is such that the volume integral of $\nabla\cdot\mathbf{A}$ across the surface remains zero, then $\mathbf{A}_\perp$ is continuous.

Similarly, that the tangential component $\mathbf{B}_\parallel$ is discontinuous by the normal integral of $\mathbf{J}$ means the tangential component $\mathbf{A}_\parallel$ is continuous unless there is a delta function sheet of $\mathbf{B}$ on the surface. Excluding this unphysical condition, we have that all three components of $\mathbf{A}$ are continuous at an interface between two media with different ϵ, μ, and σ, even across a dielectric–conductor interface on which there may be a single-layer current sheet.

The normal derivative of $\mathbf{A}_\parallel$ is not continuous if there is a skin current, and is determined by the usual boundary conditions on $\mathbf{B}_\parallel$. The discontinuity of $\mathbf{E}_\perp$ at a surface charge density shows up in the potentials as a discontinuity in the normal component of $\nabla\phi$.

B. Conductor moving through field-free region

If a conductor is moved through a region of space where $\mathbf{E}=\mathbf{B}=0$, but $\mathbf{A}\neq 0$, there are no physical effects, for there are no forces to produce any. Since $\mathbf{A}$ is continuous across the conductor surface, $\mathbf{A}$ effectively penetrates freely through the conductor.

To an observer at rest on the conductor, $\mathbf{A}$ is changing in time at a rate $d\mathbf{A}/dt=\mathbf{v}\cdot\nabla\mathbf{A}$, where $\mathbf{v}$ is the conductor velocity. By a Lorentz transformation, the observer sees a scalar potential $\phi=-\mathbf{v}\cdot\mathbf{A}$, just such as to keep $\mathbf{E}=-\partial\mathbf{A}/\partial t-\nabla\phi$ equal to zero.

C. Shielding by a perfectly conducting enclosure

Consider a localized current distribution with its attendant vector potential with $\mathbf{B}=0$. We enclose it in a conducting shield. Two "experiments" are considered:

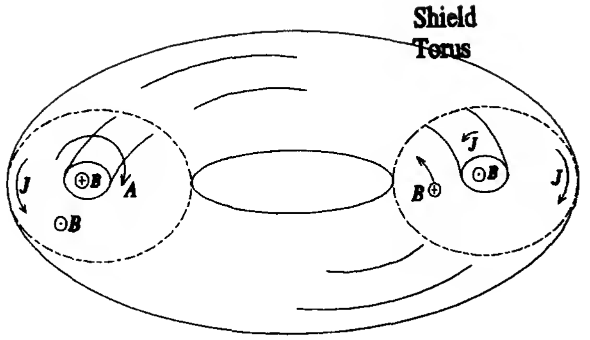

Fig. 10. Torus shielded by a larger torus.

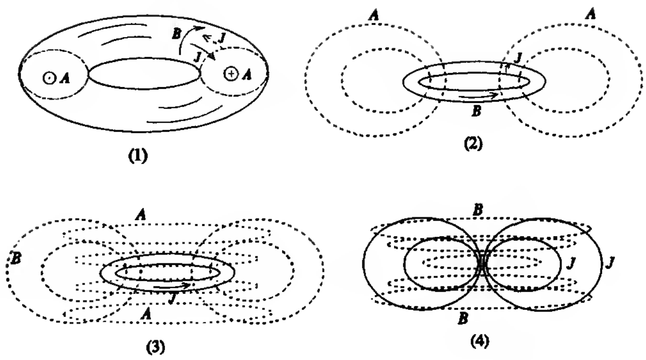

Fig. 11. Hierarchy of current distributions based on torus.

(1) The current and vector potential are pre-established and the shield box is erected around the source in the space where $\mathbf{A}\neq 0$.
(2) The shield box is constructed while the source is off, then the source is turned on.

We will choose a torus for the source.

1. Pre-established dc field

With a steady current flowing, **A** is nonzero outside. As the conductors are assembled, they move in the space through **A**. **A** passes freely through the metal pieces; there is no interaction between **A** and the conductors. We end up with the torus with its unperturbed static **A**, enclosed by a conducting box, with $\mathbf{A}\neq 0$ inside and outside the box. **A** has no way of knowing that the shield box was built.

2. dc field turned on after shield is built

In this case we erect the shield box around the dead torus, then turn the current on. The shield interrupts **A**, **E**, and **B** that are propagating out. So that concepts are not clouded by noncompatible geometries, choose for the box a concentric torus.

E, **B**, and **A** remain zero inside the shield metal. Therefore, on the inner surface of the shield, **E**, **A**, and $\mathbf{B}_\perp$ also remain zero. Surface currents which terminate $\mathbf{B}_\parallel$ are generated on the inner shield surface. The space between the driving torus and the shield torus is filled with **B** and **A**.

If I is suddenly turned on and then held constant, the final static configuration is as shown in Fig. 10. A skin current flows on the inner surface of the shield, in the opposite direction of the driving current. An azimuthal **B** field persists within the shield box, opposite the main **B** in the driving torus, so that the total flux through the large torus arm stays zero. The static **A** is in the toroidal direction between the two tori, but is 0 at the shield. **A**, **E**, and **B** all vanish outside the shield. The box shields **A** as well as the fields.

There is thus a significant difference according as the shielding enclosure is constructed after or before the driving current is activated. In the former case, the static **A** is nonzero outside the shield, but in the latter case it stays zero.

If the enclosing box is a finite conductor, currents and fields diffuse into the metal and penetrate the shield of thickness d after a time of order

$$t_d=\mu_0\sigma d^2 \tag{8.1}$$

($t_d\approx 180\ \mu\text{s}$ for 2 mm thick Al). After this time, **A** will appear outside the "shield box," and the exterior dc vector potential will have been established.

IX. VARIATIONS ON A THEME

The observations in Sec. II lead to a physically meaningful hierarchy of static current configurations.

A. Hierarchy of J, B, A

Consider the current distribution of the torus, Fig. 1, redrawn in Fig. 11(2). Call its current $\mathbf{J}_2$, field $\mathbf{B}_2$, and vector potential $\mathbf{A}_2$.

Construct a new current distribution $\mathbf{J}_3=\mathbf{B}_2$ having only an azimuthal ($\hat{\varphi}$) component. This is the current of an ordinary current loop (except that $\mathbf{J}_3$ drops off as $1/\rho$ within the loop). Its field $\mathbf{B}_3$ will be $\mathbf{B}_3=\mathbf{A}_2$. Its vector potential must be computed anew. $\mathbf{A}_3$ is also azimuthal, and lines of $\mathbf{A}_3$ form circles about the z axis in space, as sketched in Fig. 11(3).

Continuing, construct $\mathbf{J}_4=\mathbf{B}_3$. This current distribution would be that of the discharge current of a battery immersed in a partially conducting fluid. The associated magnetic field is $\mathbf{B}_4=\mathbf{A}_3$, being circles about the axis, encircling lines of $\mathbf{J}_4$. The vector potential $\mathbf{A}_4$ would have a field line pattern close to that of $\mathbf{J}_4$.

Proceeding in the opposite direction, from configuration (2) invent a current distribution $\mathbf{J}_1\propto\nabla\times\mathbf{J}_2$ whose field $\mathbf{B}_1=\mathbf{J}_2$, and vector potential $\mathbf{A}_1=\mathbf{B}_2$. This is the azimuthal double-layer current sheet discussed in Sec. IV, whose field and vector potential vanish outside the torus. It is sketched in Fig. 11(1).

One could go one step further and construct four alternating layers of toroidal currents, $\mathbf{J}_0$, confining $\mathbf{B}_0$ to be the same double layer as $\mathbf{J}_1$. $\mathbf{A}_0$ would be like $\mathbf{B}_1$, confined to a thin toroidal sheet, the same as $\mathbf{J}_2$. **A**, **B**, and **J** vanish inside and outside the torus.

Developments similar to this can be based on a common cylindrical solenoid or any other current distribution. The hierarchy is summarized by

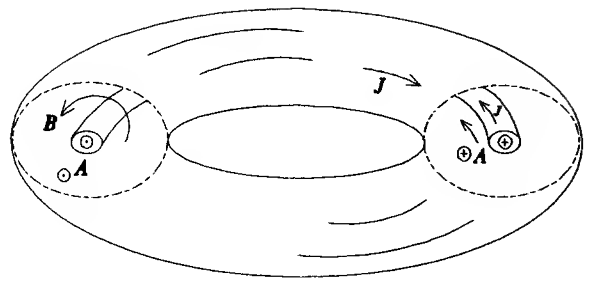

Fig. 12. Toroidal coax line.

$$\begin{array}{lllllll}
\mathbf{J}_m\cdots = & \nabla\times\mathbf{J}_1 & & & & & \\
\mathbf{B}_m\cdots = & \mathbf{J}_1 & = & \nabla\times\mathbf{J}_2 & & & \\
\mathbf{A}_m\cdots = & \mathbf{B}_1 & = & \mathbf{J}_2 & = & \nabla\times\mathbf{J}_3 & \\
 & \mathbf{A}_1 & = & \mathbf{B}_2 & = & \mathbf{J}_3 & = \ldots\nabla\times\mathbf{J}_n \\
 & & & \mathbf{A}_2 & = & \mathbf{B}_3 & = \ldots\mathbf{J}_n \\
 & & & & & \mathbf{A}_3 & = \ldots\mathbf{B}_n \\
 & & & & & & \ldots\mathbf{A}_n
\end{array}$$

which is written more succinctly in Eq. (2.11).

B. Toroidal coax

In the configuration of Fig. 11(1), one can separate the inner current sheet from the outer one, collapsing it to an inner wire ring concentric with the torus limb, as in Fig. 12. One has a toroidal coax line. **J** and **A** are azimuthal, **A** being nonzero both inside the inner conductor, and between the inner and outer one. **B** is in the "toroidal" direction, confined to between the inner conductor and outer "shield," being the magnetic field of an ordinary coax line. **A** vanishes outside the entire configuration and inside at the outer conductor.

As the inner wire is an ordinary current loop, this configuration is a "shielded current loop." The shield is an ordinary (perfect) conductor and shields both **A** and **B**. The same is true for ordinary straight coax cables.

X. A SYMMETRY OF MAXWELL'S EQUATIONS

It is interesting to inquire whether the magnetostatic observations of Sec. II on the **J**, **B**, **A** hierarchy can be extended to time varying currents and fields.

Due to the general relations $\mathbf{B}=\nabla\times\mathbf{A}$, $\nabla\cdot\mathbf{A}=\mathbf{0}$, and the static relation $\mathbf{J}=\nabla\times\mathbf{B}$, as well as $\nabla\cdot\mathbf{B}=0$, we have for the arbitrary time dependent case,

> The vector potential $\mathbf{A}_2(\mathbf{r},t)$ of any current distribution $\mathbf{J}_2(\mathbf{r},t)$ with field $\mathbf{B}_2(\mathbf{r},t)$ is the same as the instantaneous static magnetic *field* $\mathbf{B}_1(\mathbf{r},t)$ of a current $\mathbf{J}_1(\mathbf{r},t)$ instantaneously equal to $\mathbf{B}_2(\mathbf{r},t)$.

Here $\mathbf{A}_2$ is in the Coulomb gauge. It appears this rule cannot be extended to develop a hierarchy of **J**, **B**, **A** configurations in parallel with the static case since the displacement current destroys the analogy. But a closely related hierarchy still holds.

In the Lorentz gauge Maxwell's equations are

$$\mathbf{B}=\nabla\times\mathbf{A},\quad \mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}-\nabla\phi,$$
$$\Box\mathbf{A}=-\mu_0\mathbf{J},\quad \Box\phi=-\frac{\rho}{\epsilon_0}. \tag{10.1}$$

Let these hold for a current $\mathbf{J}_1$ with fields $\mathbf{E}_1$, $\mathbf{B}_1$, and potentials $\mathbf{A}_1$, ϕ_1.

Form the new current

$$\mathbf{J}_2=\nabla\times\mathbf{J}_1,\quad \nabla\cdot\mathbf{J}_2=0. \tag{10.2}$$

Clearly, its potentials are

$$\mathbf{A}_2=\nabla\times\mathbf{A}_1=\mathbf{B}_1,$$
$$\phi_2=0. \tag{10.3}$$

The associated fields are

$$\mathbf{E}_2=-\frac{\partial\mathbf{A}_2}{\partial t}=-\frac{\partial\mathbf{B}_1}{\partial t}=\nabla\times\mathbf{E}_1,$$
$$\mathbf{B}_2=\nabla\times\mathbf{A}_2=\nabla\times\mathbf{B}_1=\mu_0\left(\mathbf{J}_1+\epsilon_0\frac{\partial\mathbf{E}_1}{\partial t}\right). \tag{10.4}$$

In summary,

$$\begin{pmatrix}\mathbf{J}\\ \mathbf{B}\\ \mathbf{E}\\ \mathbf{A}\end{pmatrix}_{(2)}=\nabla\times\begin{pmatrix}\mathbf{J}\\ \mathbf{B}\\ \mathbf{E}\\ \mathbf{A}\end{pmatrix}_{(1)}, \tag{10.5}$$

a complete parallel to the magnetostatic case (2.11). Even for full time dependence the curl of one set of currents, fields, and potentials is another set, so that a solution for one **J** provides the solution for other problems. That this should be true for Maxwell's equations is less obvious than for magnetostatics, although it follows as well simply by taking the curl of Maxwell's field equations. It differs from magnetostatics in that $\mathbf{E}\neq 0$, and $\nabla\times\mathbf{B}$ is no longer **J**, so the hierarchy is not as tight. This is made more explicit by writing Eq. (10.5) as

$$\begin{pmatrix}\mathbf{J}_2\\ \mathbf{B}_2\\ \mathbf{E}_2\\ \mathbf{A}_2\end{pmatrix}=\begin{pmatrix}\nabla\times\mathbf{J}_1\\ \mathbf{J}_1\\ 0\\ \mathbf{B}_1\end{pmatrix}+\begin{pmatrix}0\\ \partial\mathbf{E}_1/\partial t\\ -\partial\mathbf{B}_1/\partial t\\ 0\end{pmatrix}, \tag{10.6}$$

in which we have dropped factors of μ_0 and ϵ_0 in the second line for simplicity. A problem-specific numerical coefficient with dimension of length should also appear multiplying the right-hand sides of Eqs. (10.5) and (10.6). Based on any given $\mathbf{J}(\mathbf{r},t)$ one can develop a hierarchy of physical current and field distributions in parallel with the magnetostatics case.

A. Fields of a rotating torus

One instance in which this symmetry is useful is in computing the fields of a rotating torus.

Let a dc current flow in the torus windings of Fig. 1. Suppose the current is delivered to the windings by slip rings so the torus is free to rotate about, say, the y axis, maintaining its dc current. (Or we could imagine a superconducting arrangement, with no need to drive the currents once established.) When stationary, there is a vector potential, but no fields, outside. When the torus is spinning, are there external fields?

Clearly there are, for to an external observer the static $\mathbf{A}$ also rotates, producing nonzero $\mathbf{E}=-\partial\mathbf{A}/\partial t$, without a canceling $-\nabla\phi$. Since $\partial\mathbf{E}/\partial t$ is also nonzero, $\nabla\times\mathbf{B}=(1/c^2)\partial\mathbf{E}/\partial t\neq 0$, and therefore $\mathbf{B}$ does not vanish outside. An oscillating EMF is generated around a fixed loop in, say, the $x=0$ plane that encircles the torus limb in its original position. A rotating torus carrying a dc current possesses nonzero near zone fields. It also radiates. The question is how to calculate the fields. The symmetry discussed above is quite powerful in this regard, and makes the task easy.

The surface current density on the torus is the curl of the current density of an ordinary current loop with the same dimensions. According to Eq. (10.5) then, the exact fields and potential of the rotating torus are precisely the curl of the corresponding fields and potential of a spinning current loop. These latter fields are not difficult to compute. Here we sketch only the radiation fields.

More precisely, a torus wound with N turns carrying total current $I=i_w N$, has a current density $\mathbf{J}$ proportional to the curl of the current density $\mathbf{J}'$ of a loop carrying current $I'=\pi a^2 J'$:

$$\mathbf{J}=\kappa\nabla\times\mathbf{J}', \quad \kappa=\frac{a^2}{2b}\frac{I}{I'}. \tag{10.7}$$

The coefficient κ should appear multiplying the right-hand side of (10.5) and (10.6).

A loop of radius b, of thin wire with radius $a\ll b$, carrying current I' has a magnetic moment $m=\pi b^2 I'$. When rotating with angular velocity ω in the positive sense about $\hat{y}$ its components are

$$m_z=m\cos\omega t, \quad m_x=m\sin\omega t. \tag{10.8}$$

Each of these is in turn the magnetic moment of a stationary loop with oscillating current. Each radiated electric field is of the standard form, and the total is the superposition of the two:

$$\mathbf{E}_{\text{rad}}^{\text{loop}}=\mathbf{E}_{\text{rad}}^{(x)}+\mathbf{E}_{\text{rad}}^{(z)},$$

$$\mathbf{E}_{\text{rad}}^{(x)}=-i\frac{\mu_0}{4\pi}\frac{\omega^2 m}{cr}e^{i(kr-\omega t)}\sin\theta_x\hat{\varphi}_x, \tag{10.9}$$

$$\mathbf{E}_{\text{rad}}^{(z)}=-\frac{\mu_0}{4\pi}\frac{\omega^2 m}{cr}e^{i(kr-\omega t)}\sin\theta_z\hat{\varphi}_z,$$

where the real part is understood. Here θ_z and φ_z are the usual polar and azimuthal angles in spherical coordinates defined on the z axis, and θ_x and φ_x are those of a spherical system defined on the x axis. The magnetic field is

$$\mathbf{B}_{\text{rad}}^{\text{loop}}=\frac{1}{c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{rad}}^{\text{loop}}. \tag{10.10}$$

For these, as for any radiation fields,

$$\nabla\times\mathbf{E}_{\text{rad}}=-\frac{\partial\mathbf{B}_{\text{rad}}}{\partial t}=i\omega\mathbf{B}_{\text{rad}}=i\frac{\omega}{c}\hat{\mathbf{r}}\times\mathbf{E}_{\text{rad}}. \tag{10.11}$$

Then, using $\hat{r}\times\hat{\varphi}_x=-\hat{\theta}_x$, etc., the radiated field of a spinning torus comes out

$$\begin{aligned}\mathbf{E}_{\text{rad}}^{\text{torus}}&=\kappa\nabla\times\mathbf{E}_{\text{rad}}^{\text{loop}}\\&=i\kappa\omega\mathbf{B}_{\text{rad}}^{\text{loop}}\\&=\frac{\mu_0 c}{4\pi}\frac{Vk^3 I}{4\pi}\frac{e^{i(kr-\omega t)}}{r}[-\sin\theta_x\hat{\theta}_x+i\sin\theta_z\hat{\theta}_z].\end{aligned} \tag{10.12}$$

κ has been replaced using $\kappa m=(\pi a^2 b/2)I=(V/4\pi)I$. The full radiation pattern can be mapped from Eq. (10.12). As for a loop, this is the field of two stationary perpendicular toruses carrying oscillating currents, each with fields of the form (6.18).

For an observer on the rotation axis ($+y$), for example, $\sin\theta_x=\sin\theta_z=1$, $\hat{\theta}_x=-\hat{x}$, $\hat{\theta}_z=-\hat{z}$, and

$$\mathbf{E}_{\text{rad}}^{\text{torus}}(x=0,y=r,z=0)=\frac{\mu_0 c}{4\pi}\frac{Vk^3 I}{4\pi}\frac{e^{i(kr-\omega t)}}{r}[\hat{x}-i\hat{z}]. \tag{10.13}$$

Radiation in the $+y$ direction is right-circularly polarized.

One can similarly compute the quasistatic fields of a spinning torus from those of a spinning loop.

The energy of the torus consists of its rotational kinetic energy plus the magnetic field energy inside; these supply the energy radiated. The kinetic energy is an artifact of the mass of material chosen for fabrication, and can in principle be made as small as desired. Therefore, the energy radiated comes from the enclosed magnetostatic field energy. Due to radiation the dc current of a spinning torus will decay.

This example has been illustrative only. One can employ the symmetry noted in this section to problems less academic than a rotating torus.

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APPENDIX A. EXACT FIELDS OF A TORUS

The complete vector potential and fields of a harmonically driven torus may be computed to arbitrary accuracy by expanding in small parameters.

The full expression for $\mathbf{A}(\mathbf{r},t)$ in the Lorentz gauge is Eq. (5.3). Taking

$$\mathbf{J}(\mathbf{r}',t')=e^{-i\omega t'}\mathbf{J}(\mathbf{r}')=e^{-i\omega t}e^{ik|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}') \tag{A1}$$

and $\mathbf{A}=\mathbf{A}_\omega(\mathbf{r})e^{-i\omega t}$, then

$$\mathbf{A}_\omega(\mathbf{r})=\frac{\mu_0}{4\pi}\int d^3r'\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}'). \tag{A2}$$

$\nabla\cdot\mathbf{A}=0$, and since $\nabla\cdot\mathbf{J}=0$, the scalar potential vanishes. We consider only $r>b$ and $kb<1$, but expressions will be valid in the near zone $kr<1$, or far zone $kr>1$. When $kb\gtrsim 1$, azimuthal asymmetries arise due to propagation time delays around the torus when driven at one point. We do not take into account these asymmetries.

The dependence on r and r' is separated using the standard expansion

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}=ik\sum_{\ell=0}^{\infty}(2\ell+1)j_\ell(kr')h_\ell^{(1)}(kr)P_\ell(\mu), \quad r>r', \tag{A3}$$

where j_ℓ is the spherical Bessel function, P_ℓ is the Legendre polynomial, and $\mu=\cos(\mathbf{r},\mathbf{r}')$. Powers of kr are explicit because the Hankel function

$$h_\ell^{(1)}(kr)=\frac{1}{i^\ell}\frac{e^{ikr}}{ikr}\sum_{q=0}^{\ell}\frac{(\ell+q)!}{q!(\ell-q)!}\left(\frac{i}{2kr}\right)^q. \tag{A4}$$

Then,

$$\mathbf{A}_\omega=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\sum_{\ell=0}^{\infty}\mathbf{S}_\ell(k)\sum_{q=0}^{\ell}f_{\ell q}\left(\frac{1}{kr}\right)^q, \tag{A5}$$

where

$$\mathbf{S}_\ell(k)=\int d^3r' j_\ell(kr')P_\ell(\mu)\mathbf{J}(\mathbf{r}') \tag{A6}$$

characterizes the source and observer angle, and

$$f_{\ell q}=\frac{(2\ell+1)}{i^{\ell-q}}\frac{(\ell+q)!}{2^q q!(\ell-q)!}. \tag{A7}$$

Since for small argument $j_\ell(kr')=(kr')^\ell/(2\ell+1)!!$, Equation (A6) shows

$$\mathbf{S}_\ell(k)=(kb)^\ell\mathbf{G}_\ell, \quad \ell>0 \tag{A8}$$

with $\mathbf{G}_\ell=\mathbf{G}_\ell(k)$ independent of k to lowest order. The only exception is $\mathbf{S}_0\propto(kb)^2$. Using this in (A5) and interchanging summations gives

$$\mathbf{A}_\omega=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\mathbf{S}_0+\sum_{q=0}^{\infty}\sum_{\ell=q}^{\infty}{}'\mathbf{G}_\ell f_{\ell q}\frac{(kb)^\ell}{(kr)^q}\right\}, \tag{A9}$$

where the prime means the $\ell=0$ term is deleted from the sum. Now setting $\ell=q+j$,

$$\mathbf{A}_\omega=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\mathbf{S}_0+\sum_{q=0}^{\infty}\left[\sum_{j=0}^{\infty}{}'\mathbf{G}_{q+j}f_{q+j,q}(kb)^j\right]\left(\frac{b}{r}\right)^q\right\}. \tag{A10}$$

This is an explicit series in powers of b/r with coefficients that are rapidly converging series in kb.

It is not difficult to show that the combination of azimuthal symmetry and reflection symmetry in the $z=0$ plane implies

$$\mathbf{S}_\ell(k)=\mathbf{G}_\ell(k)=0, \quad \ell \text{ odd} \tag{A11}$$

so that only even indices survive in (A10):

$$\mathbf{A}_\omega=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\mathbf{S}_0+\sum_{\ell=1}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,0}(kb)^{2\ell}+\frac{b}{r}\sum_{\ell=1}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,1}(kb)^{2\ell-1}+\left(\frac{b}{r}\right)^2\sum_{\ell=1}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,2}(kb)^{2\ell-2}\right.$$
$$\left.+\left(\frac{b}{r}\right)^3\sum_{\ell=2}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,3}(kb)^{2\ell-3}+\left(\frac{b}{r}\right)^4\sum_{\ell=2}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,4}(kb)^{2\ell-4}+\left(\frac{b}{r}\right)^5\sum_{\ell=3}^{\infty}\mathbf{G}_{2\ell}f_{2\ell,5}(kb)^{2\ell-5}+\dots\right\}. \tag{A12}$$

$\mathbf{A}_\omega$ contains all powers of $1/r$, but as $k\to 0$ the static vector potential

$$\mathbf{A}_0=\frac{\mu_0}{4\pi}\frac{1}{r}\left\{\left(\frac{b}{r}\right)^2\mathbf{G}_2 f_{22}+\left(\frac{b}{r}\right)^4\mathbf{G}_4 f_{44}+\cdots\right\} \tag{A13}$$

contains only odd inverse powers of r.

The first few terms of (A12) are

$$\mathbf{A}_\omega=\frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\left\{f_{00}\mathbf{S}_0+f_{20}\mathbf{G}_2(kb)^2+\frac{b}{r}[f_{21}\mathbf{G}_2(kb)\right.$$
$$+f_{41}\mathbf{G}_4(kb)^3]+\left(\frac{b}{r}\right)^2[f_{22}\mathbf{G}_2+f_{42}\mathbf{G}_4(kb)^2]$$
$$+\left(\frac{b}{r}\right)^3[f_{43}\mathbf{G}_4(kb)+f_{63}\mathbf{G}_6(kb)^3]+O((kb)^4)$$
$$\left.+O((b/r)^4)\right\}. \tag{A14}$$

In evaluating the higher powers, the k dependence of $\mathbf{G}_\ell$ must be included. $\mathbf{G}_2$ has already been evaluated, since according to (A13) it is determined by the static $1/r^3$ potential, which was obtained in Sec. III. Comparing the first term of (A13) with (3.5) gives

$$f_{22}\mathbf{G}_2=\frac{\pi a^2 I}{2b}(2\cos\theta\hat{r}+\sin\theta\hat{\theta}), \tag{A15}$$

and direct evaluation shows

$$\mathbf{S}_0=(kb)^2\mathbf{G}_0=(kb)^2\frac{\pi a^2 I}{3b}(\cos\theta\hat{r}-\sin\theta\hat{\theta}). \tag{A16}$$

Then, taking the f's from (A7), one gets

$$\mathbf{A}(t)=\frac{\mu_0}{4\pi}\frac{VI}{4\pi r^3}e^{-i\omega(t-r/c)}[(1-ikr)(2\cos\theta\hat{r}$$
$$+\sin\theta\hat{\theta})-k^2r^2\sin\theta\hat{\theta}], \tag{A17}$$

plus smaller terms of higher order in b/r and kb. Here $V=2\pi^2a^2b$=torus volume.

The first term (1) in square brackets is the lowest order static vector potential, separately obtained in Eq. (3.5). The ikr term is the "inductive" potential. The term in k^2r^2 is the radiated potential, previously obtained in Eq. (6.17).

The fields to the same order are

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t}=i\omega\mathbf{A}, \tag{A18}$$

having terms behaving as $\omega I/r^3$, $\omega^2 I/r^2$, and $\omega^3 I/r$; and

$$\mathbf{B}=\nabla\times\mathbf{A}=\frac{\mu_0}{4\pi}\frac{VI}{4\pi}e^{-i\omega(t-r/c)}\frac{k^2}{r^2}[1-ikr]\sin\theta\hat{\varphi}, \tag{A19}$$

behaving as $\omega^2 I/r^2$ and $\omega^3 I/r$.

In the near zone, the quasistatic magnetic and electric fields, given in (5.6), are related by Eq. (5.7).

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Teaching physics with 670 nm diode lasers—construction of stabilized lasers and lithium cells

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We describe the construction and operation of stabilized 670 nm diode lasers for use in undergraduate teaching labs. Because they emit low-power visible radiation, 670 nm lasers are safe and aesthetically pleasing, and thus are an attractive alternative to near-infrared diode lasers in the undergraduate laboratory. We also describe the fabrication of a robust and reliable lithium atomic vapor cell, which can be used with the 670 nm diode lasers to perform a variety of atomic physics experiments. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

As inexpensive sources of narrowband tunable coherent light, semiconductor diode lasers are becoming important research tools with widespread applications in modern academic and industrial laboratories. Because of their relatively low cost (compared to other tunable laser sources) they are also well-suited for incorporation into undergraduate teaching labs. By scanning the laser emission over atomic or molecular resonance lines, a variety of interesting and fundamental physics experiments can be performed.

A recent paper by MacAdam *et al.*[1] describes several techniques for constructing and operating stabilized diode lasers for the undergraduate laboratory. These lasers operate in the near infrared, at 780 or 852 nm, and can be tuned to scan over various rubidium or cesium resonance lines, respectively. Unfortunately, radiation at these wavelengths is almost completely invisible to the human eye, which creates several problems when incorporating near-IR lasers into undergraduate labs. A primary concern is safety, since even a milliwatt of laser radiation can cause irreparable eye damage.

